MELLIN TRANSFORM AND LAPLACE'S EQUATION

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The Mellin transform of a function $f(r)$ is defined by

$$M[s;f] \equiv \int_0^\infty f(\rho)\rho^{s-1}d\rho \tag{1}$$

with inverse transform

$$f(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M[i\omega;f]r^{-i\omega}d\omega \tag{2}$$

We can use the Mellin transform to give solutions to Laplace's equation which has the form in two-dimensional polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0 \tag{3}$$

One solution that uses the Mellin transform is

$$\phi(r,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M[i\omega,f]}{\sinh\omega\theta_0}r^{-i\omega}\sinh\omega\theta\, d\omega \tag{4}$$

where θ_0 is a constant angle. Using Maple for the derivatives in 3 we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M[i\omega,f]r^{-i\omega}}{\sinh\omega\theta_0}\frac{\sinh\omega\theta}{r^2}d\omega \tag{5}$$

$$\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M[i\omega,f]r^{-i\omega}}{\sinh\omega\theta_0}\frac{\sinh\omega\theta}{r^2}d\omega = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) \tag{6}$$

Thus 3 is satisfied.

The solution satisfies the boundary conditions on the wedge $0<\theta<\theta_0$ given by

$$\begin{aligned}\phi(r,0) &= 0\\ \phi(r,\theta_0) &= f(r)\end{aligned} \tag{7}$$

as we can verify from 4:

$$\phi(r,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M[i\omega,f]}{\sinh\omega\theta_0}r^{-i\omega}\sinh 0\, d\omega = 0 \tag{8}$$

$$\phi(r,\theta_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M[i\omega,f]}{\sinh\omega\theta_0}r^{-i\omega}\sinh\omega\theta_0 d\omega \tag{9}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} M\left[i\omega;f\right] r^{-i\omega} d\omega \tag{10}$$
$$= f\left(r\right) \tag{11}$$

using 2.

Another solution that satisfies the boundary conditions

$$\begin{aligned} \phi\left(r,0\right) &= g\left(r\right) \\ \phi\left(r,\theta_0\right) &= 0 \end{aligned} \tag{12}$$

is given by modifying the argument of the upper $\sinh\omega\theta$ factor and taking the negative:

$$\phi\left(r,\theta\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{M\left[i\omega,g\right]}{\sinh\omega\theta_0} r^{-i\omega}\sinh\omega\left(\theta-\theta_0\right)\, d\omega \tag{13}$$

We can verify that this is still a solution by calculating the derivatives in 3 again, which gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M\left[i\omega,g\right]r^{-i\omega}}{\sinh\omega\theta_0}\frac{\sinh\omega\left(\theta-\theta_0\right)}{r^2}d\omega \tag{14}$$

$$\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M\left[i\omega,g\right]r^{-i\omega}}{\sinh\omega\theta_0}\frac{\sinh\omega\left(\theta-\theta_0\right)}{r^2}d\omega = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) \tag{15}$$

For the boundary conditions, we have

$$\phi\left(r,0\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{M\left[i\omega,g\right]}{\sinh\omega\theta_0} r^{-i\omega}\sinh\left(-\omega\theta_0\right)d\omega \tag{16}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{M\left[i\omega,g\right]}{\sinh\omega\theta_0} r^{-i\omega}\sinh\left(\omega\theta_0\right)d\omega \tag{17}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} M\left[i\omega;g\right] r^{-i\omega} d\omega \tag{18}$$
$$= g\left(r\right) \tag{19}$$
$$\phi\left(r,\theta_0\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{M\left[i\omega,g\right]}{\sinh\omega\theta_0} r^{-i\omega}\sinh 0\, d\omega = 0 \tag{20}$$

We can combine these two solutions to give

$$\phi\left(r,\theta\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{M\left[i\omega,f\right]}{\sinh\omega\theta_0} r^{-i\omega}\sinh\omega\theta\, d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{M\left[i\omega,g\right]}{\sinh\omega\theta_0} r^{-i\omega}\sinh\omega\left(\theta-\theta_0\right)\, d\omega \tag{21}$$

This solution satisfies the boundary conditions

$$\begin{aligned}\phi(r,0) &= g(r)\\ \phi(r,\theta_0) &= f(r)\end{aligned} \tag{22}$$

Finally, we consider the boundary conditions

$$\begin{aligned}\frac{\partial\phi(r,0)}{\partial\theta} &= 0\\ \phi(r,\theta_0) &= f(r)\end{aligned} \tag{23}$$

A solution satisfying these conditions is

$$\phi(r,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M[i\omega,f]}{\cosh\omega\theta_0}r^{-i\omega}\cosh\omega\theta\; d\omega \tag{24}$$

We again verify that it satisfies 3:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M[i\omega,f]\,r^{-i\omega}}{\cosh\omega\theta_0}\frac{\cosh\omega\theta}{r^2}d\omega \tag{25}$$

$$\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M[i\omega,f]\,r^{-i\omega}}{\cosh\omega\theta_0}\frac{\cosh\omega\theta}{r^2}d\omega = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) \tag{26}$$

We check the boundary conditions from 24:

$$\frac{\partial\phi}{\partial\theta} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M[i\omega,f]}{\cosh\omega\theta_0}r^{-i\omega}\omega\sinh\omega\theta\; d\omega \tag{27}$$

so

$$\frac{\partial\phi(r,0)}{\partial\theta} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M[i\omega,f]}{\cosh\omega\theta_0}r^{-i\omega}\omega\sinh 0\; d\omega = 0 \tag{28}$$

and

$$\phi(r,\theta_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{M[i\omega,f]}{\cosh\omega\theta_0}r^{-i\omega}\cosh\omega\theta_0\; d\omega \tag{29}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}M[i\omega;f]\,r^{-i\omega}d\omega \tag{30}$$

$$= f(r) \tag{31}$$